Slow-light solitons: influence of relaxation

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We have applied the transformation of the slow light equations to Liouville theory that we developed in our previous work, to study the influence of relaxation on the soliton dynamics. We solved the problem of the soliton dynamics in the presence of relaxation and found that the spontaneous emission from the upper atomic level is strongly suppressed. Our solution proves that the spatial shape of the soliton is well preserved even if the relaxation time is much shorter than the soliton time length. This fact is of great importance for applications of the slow-light soliton concept in optical information processing. We also demonstrate that the relaxation plays a role of resistance to the soliton motion and slows the soliton down even if the controlling field is constant.

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The development of modern methods of optical signal manipulation and control opens wide perspectives for application of the light in classical and quantum computation. One of such methods of optical signal processing is based on the effect of electromagnetically induced transparency (EIT) [1], which allows for slowing the light down by many orders of magnitude and even bringing it to a complete halt [2, 3, 4]. The major advantage of this methods is that the velocity of the optical signal is effectively controlled by an auxiliary laser [5, 6, 7]. In the linear regime of operation when the intensity of the controlling field is significantly larger than the intensity of the signal, there are some important constraints imposed on the parameters of the signal [1]. These constraints result from the presence of strong optical relaxation in the medium. The medium is typically a gas of alkali atoms whose electronic structure relevant to the EIT effect can be schematically described by the three-level Λ model (see Fig. 1). The relaxation results from spontaneous transitions of the atoms from the excited upper energy level $|3\rangle$. Usually, such transitions may occur not only to the levels $|1\rangle$,$|2\rangle$ but also to other lower levels, which are not included into consideration for the sake of simplicity. In any case, the relaxation destroys optical coherence of the signal and therefore must be accounted for.

The atom-field interaction in the system with EIT is substantially nonlinear. In our previous works we provided a nonlinear solution of the EIT problem called the slow-light soliton [8]. We also demonstrated that the dynamics of slow-light solitons strongly depends on the form of the controlling field and the solitons can be effectively manipulated concerning their space-time dynamics [9, 10]. In our present work we study the dynamics of the solitons in the presence of the relaxation. We note that in the framework of the nonlinear theory there might exist different approaches to effectively suppress the influence of relaxation. Here, we demonstrate that appropriately choosing the parameters of the controlling field we can reduce the influence of the relaxation on the slow-light solitons by several orders of magnitude. Our results are found to be in a good agreement with the experiments on EIT.

Within the slowly varying amplitude and phase approximation (SVEPA) the Maxwell equations for the Rabi-frequencies are reduced to the the well-known equations governing the dynamics of the atom-field system, viz.

$$\partial_\zeta \Omega_a = i\nu_0\, \psi_3\psi_1^*,\ \partial_\zeta \Omega_b = i\nu_0\, \psi_3\psi_2^*. \tag{1}$$

Here $\zeta = z/c, \tau = t - z/c$. The Schrodinger equation for

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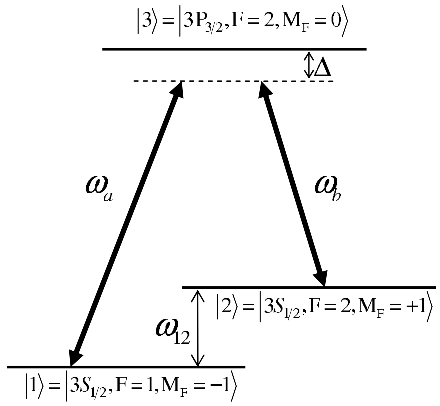

FIG. 1: The Λ-scheme for sodium atoms.

the amplitudes $\psi_{1,2,3}$ of atomic wave function reads

$$\begin{aligned} \partial_\tau \psi_1 &= \frac{i}{2}\Omega_a^*\,\psi_3; \\ \partial_\tau \psi_2 &= \frac{i}{2}\Omega_b^*\,\psi_3; \\ \partial_\tau \psi_3 &= -(i\,\Delta + \frac{\gamma}{2})\,\psi_3 + \frac{i}{2}(\Omega_a\psi_1 + \Omega_b\psi_2). \end{aligned} \tag{2}$$

By excluding the amplitudes of the lower levels the system of Eqs.(1), (2) can be transformed to the following form

$$\tfrac{1}{\psi_3^*}\partial_\tau \tfrac{1}{\psi_3}\partial_\zeta \Omega_a = \tfrac{\nu_0}{2}\Omega_a; \tag{3}$$
$$\tfrac{1}{\psi_3^*}\partial_\tau \tfrac{1}{\psi_3}\partial_\zeta \Omega_b = \tfrac{\nu_0}{2}\Omega_b; \tag{4}$$
$$\partial_\tau |\psi_3|^2 = -\gamma\,|\psi_3|^2 - \tfrac{1}{2\nu_0}\partial_\zeta(|\Omega_a|^2 + |\Omega_b|^2). \tag{5}$$
$$\partial_\tau \varphi_3 = -\Delta + \tfrac{1}{2\nu_0\,|\psi_3|^2}(|\Omega_a|^2\,\partial_\zeta\varphi_a + |\Omega_b|^2\,\partial_\zeta\varphi_b). \tag{6}$$

Here, $\varphi_{a,b,3}$ are respectively the phases of the fields $\Omega_{a,b}$ and the amplitude of the excited state ψ_3. Notice that the equations Eqs. (3),(4) are the wave equations for the fields in curvilinear coordinates with the metric dependent on the amplitude ψ_3.

In the typical dynamics scenario, the atoms are initially prepared on the lowest energy level $|1\rangle$, the probe field is absent while the controlling field is constant. Notice that the state $|1\rangle$ is a dark state for the controlling field. Thus, the initial state of the system reads

$$\Omega_a(0,\zeta) = 0,\ \Omega_b(0,\zeta) = \Omega_0,\ |\psi_{at}(0,\zeta)\rangle = |1\rangle. \tag{7}$$

In fact, this is the simplest solution of the equations Eqs.(3),(4),(5),(6).

I. RELAXATION

From the results of linear and nonlinear theories of electromagnetically induced transparency (EIT) [9, 11, 12, 13, 14, 15] we infered a physically plausible assumption that the population of the excited state $|3\rangle$ is proportional to the amplitude of the field in the probe channel, which we denote as Ω_a. In our previous works we demonstrated that a slow-light soliton in the system without relaxation is obtained by taking $\psi_3 = -\frac{1}{2|\lambda-\Delta|}\Omega_a$. Here, λ is an arbitrary parameter limited from below by the condition $|\psi_3| \le 1$. In order to take into account the relaxation we propose a more general relation between the amplitude of atomic upper level $|3\rangle$ and the amplitude Ω_a of the slow-light soliton. The population of the upper level although very small should remain nonzero in order to preserve necessary coupling between the probe and controlling field. At the same time, the relaxation should manifest itself in the form of some effective damping of the probe field. Based on such consideration we found the following relation between the amplitudes:

$$e^{\alpha(\tau)}\psi_3 = -\frac{1}{2|\lambda-\Delta|}\Omega_a, \tag{8}$$

where $\alpha(\tau)$ is a first order correction to the exact slow-light soliton solution. It will be shown below that $\alpha(\tau)$ is always negative. Let us introduce convenient notations

$$|\Omega_a| \equiv e^{-\rho},\ \Omega_b \equiv \eta,\ \tilde{\rho} \equiv \rho + \alpha,\ k = \frac{\nu_0}{8|\lambda-\Delta|^2}.$$

The first three equations of the system Eq.(**??**) take then the following form:

$$\partial_\tau\alpha\ \partial_\zeta\tilde{\rho} + \partial_{\zeta\tau}\tilde{\rho} = -k\,e^{-2\tilde{\rho}} \tag{9}$$
$$\partial_{\zeta\tau}\eta + \partial_\tau\tilde{\rho}\ \partial_\zeta\eta = k\,e^{-2\tilde{\rho}}\,\eta, \tag{10}$$
$$4k(\partial_\tau + \gamma)\,e^{-2\tilde{\rho}} = -\partial_\zeta\left(\eta^2 + e^{-2\rho}\right). \tag{11}$$

The last equation for the phase φ_3 can easily be integrated after we solved the first three.

Assuming that the effective relaxation described by α varies slowly in time τ and therefore neglecting the first term in Eq.(9) we can find a solution of the system of equations Eqs.(9), (10), (11). We will provide a rigorous description of this approximation after we construct the solution in the next section.

II. SOLUTION

We assume that $\partial_\tau\alpha$ is small as compared to the r.h.s. of the equation Eq.(9), and neglect the first term in the l.h.s of this equation. The equation then transforms into the well known Liouville equation, viz.

$$\partial_{\zeta\tilde{\tau}}\tilde{\rho} = -k\,e^{-2\tilde{\rho}}, \tag{12}$$

whose general solution is readily available:

$$\rho = -\frac{1}{2}\log\left[\frac{\frac{1}{k}e^{2\alpha}\partial_\zeta\,A_+(\zeta)\partial_\tau\,A_-(\tau)}{(1-A_+A_-)^2}\right]. \tag{13}$$

Here $A_+(\zeta), A_-(\tau)$ are arbitrary functions. To obtain the solution of the whole system including Eqs.(10) and

(11) we specify these arbitrary functions as follows:

$$A_+(\zeta) = -\exp[-8\varepsilon_0 k\zeta], \tag{14}$$

$$A_-(\tau) = \exp\left[2\varepsilon_0 \int \frac{e^{2\alpha(\tau)}}{p(\tau)^2+1} d\tau\right], \tag{15}$$

$$\eta = -2p\,\partial_\tau \rho + 2\partial_\tau p - 2p\,\partial_\tau \alpha, \tag{16}$$

$$\partial_\tau \alpha(\tau) = -\frac{\gamma/2}{p(\tau)^2+1}. \tag{17}$$

$p(\tau)$ is an arbitrary function describing the controlling field $\Omega(\tau)$, while $\lambda = i\,\varepsilon_0$. We note from Eq.(17) that the correction α to exact solution without relaxation obtained in our previous papers vanishes for $\gamma = 0$ as expected.

So, the fields read

$$\Omega_a(\tau,\zeta) = \frac{2\varepsilon_0 e^{2\alpha}}{\sqrt{p(\tau)^2+1}} \operatorname{sech}(\varphi), \tag{18}$$

$$\Omega_b(\tau,\zeta) = -\frac{2\varepsilon_0\, p(\tau) e^{2\alpha}}{p(\tau)^2+1} \tanh(\varphi) + \frac{2\partial_\tau p(\tau) - \gamma p(\tau)}{p(\tau)^2+1}$$

with the phase

$$\varphi = -4k\,\varepsilon_0\,(\zeta - \zeta_0) + \varepsilon_0 \frac{1 - e^{2\alpha(\tau)}}{\gamma}. \tag{19}$$

We chose $\alpha(0) = 0$. From the solution for the phase of slow-light soliton Eq.(19) we obtain its velocity:

$$v_g = \frac{1}{4k} \frac{e^{2\alpha}}{p(\tau)^2+1}. \tag{20}$$

III. VALIDITY OF THE APPROXIMATION

Now we will analyze the validity of our approximation of small $\partial_\tau \alpha$. The first term of Eq.(9) can be neglected if it is much smaller than $k\,e^{-2\tilde{\rho}}$. Hence we find the following condition on α:

$$|\partial_\tau \alpha| \ll \left| \frac{8\varepsilon_0 e^{2\alpha}}{(p^2+1)\sinh(2\varphi)} \right|. \tag{21}$$

This condition can be simplified with the help of Eq.(17) and we obtain

$$\frac{\gamma}{16\varepsilon_0} \ll \frac{e^{2\alpha}}{|\sinh(2\varphi)|}. \tag{22}$$

It is always fulfilled at the maximum of slow-light soliton, because $\varphi = 0$ there. The position of the maximum of the soliton is given by the following function of retarded time τ:

$$\zeta_c(\tau) = \frac{1 - e^{2\alpha(\tau)}}{4k\gamma} + \zeta_0, \tag{23}$$

where we chose $\zeta_c(0) = \zeta_0$. Hence, the phase of the soliton can be rewritten as

$$\varphi = -4k\,\varepsilon_0\,(\zeta - \zeta_c(\tau)). \tag{24}$$

From equation Eq.(22) we can find the spatial window of validity of our approximation with respect to the center of slow-light soliton $\Delta\zeta(\tau) = \zeta - \zeta_c(\tau)$ where our soliton provides correct description of the pulse shape. It reads

$$|\sinh(-8k\,\varepsilon_0\;\Delta\zeta)| \ll \frac{16\varepsilon_0}{\gamma} e^{2\alpha}. \tag{25}$$

At the initial moment of time, for $\gamma < 16\varepsilon_0$ we find $k\,\varepsilon_0\,\Delta\zeta_0 \approx \ln\left(\frac{2\varepsilon_0}{\gamma}\right)$. We will denote the full-width at half-maximum of slow-light soliton by $w_s \approx 0.66/(k|\varepsilon_0|)$. To keep our solution valid at least within the full-width at half-maximum of the soliton, i.e. $w_s = 2\Delta\zeta_0$, the parameter ε_0 should obey the following condition

$$\varepsilon_0 \geq 0.7\gamma \tag{26}$$

Notice that α is a negative monotonically decreasing function of τ. Therefore, the validity window closes down with the retarded time τ. However, the group velocity of the slow-light soliton approaches zero as τ increases and the soliton slows down to full stop (see Eq.(20)). The distance that the soliton travels until full stop is

$$\mathcal{L} \equiv \zeta_c(\infty) - \zeta_0 \leq \frac{1}{4k\gamma}. \tag{27}$$

This formula clearly shows that the maximum distance that our soliton can travel in the medium is limited by the magnitude of the relaxation constant. As stronger the relaxation is, as smaller the distance that the soliton can propagate in the medium.

IV. COMPARISON WITH EXPERIMENTS

We discussed in our previous papers that the controlling field generated by an auxiliary laser on entrance into the medium supports the propagation of the slow-light soliton. An appropriately chosen modulation of the controlling field ensures the nonlinear coupling between the soliton and atoms, while the nonlinear coupling in its turn preserves the secant shape of the soliton. However, in the ideal case when the relaxation is absent, the modulation of the controlling field has the same time length as the signal in the channel a. After the soliton has been generated, the auxiliary laser continues emitting a constant laser beam, which supports the propagation of the soliton with a constant velocity. We demonstrated that this ideal situation when the amplitude of the soliton remains constant corresponds to a constant function $p(\tau) = p_0$. If the relaxation is not zero, the amplitude and the velocity of the soliton will be decaying in time τ according to Eq.(18). The asymptotic value of the background field past the modulation supporting the soliton can be found from $\Omega_b(\tau,\zeta)$ by taking the limit $\zeta \to \infty$ at the zero moment of time:

$$\Omega_0 = \frac{(2|\varepsilon_0| - \gamma)p_0}{p_0^2+1}. \tag{28}$$

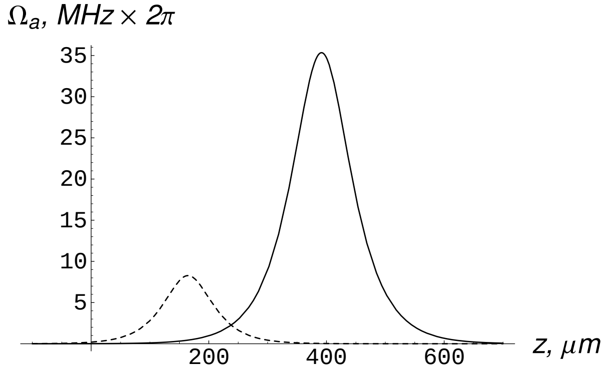

FIG. 2: Slow-light soliton decay due to the relaxation. The solid line corresponds to a reference soliton propagating in the system without relaxation, and the dashed line represents the pulse in the presence of relaxation.

Assuming that $\gamma \gg \Omega_0$ we find

$$p_0 = \frac{2|\varepsilon_0| - \gamma}{2\Omega_0} + \sqrt{\left(\frac{2|\varepsilon_0| - \gamma}{2\Omega_0}\right)^2 - 1} \simeq \frac{2|\varepsilon_0| - \gamma}{\Omega_0}. \quad (29)$$

In the absence of the relaxation, the time length of the soliton pulse on entrance into the medium is determined by the phase $\varphi_0(t) = \varepsilon_0\, t/(p_0^2 + 1)$. Hence we obtain a second condition on two arbitrary parameters p_0 and ε_0 in terms of well-defined experimental parameters Ω_0 and t_p:

$$\operatorname{sech}\left(\frac{|\varepsilon_0|}{p_0^2 + 1}\, t_p/2\right) = 0.5 \quad (30)$$

Choosing $|\varepsilon_0| = \gamma$ we verify that the conditions Eqs.(26), (28), (30) are well satisfied for experimentally feasible values of $t_p \sim 1\mu s$ and $\Omega_0 \sim 10\,\mathrm{MHz}$.

Now we will compare our solution with the experimental results reported for sodium atoms [2]. The parameters have the following values: $\gamma = 6.3 \times 10^7\ \mathrm{rad\,s^{-1}}$, $t_p = 2.5\mu s$, and $\Omega_0 = 0.56\gamma$. We solve Eqs.(28), (30) and find $p_0 \simeq 18.4$, $\varepsilon_0 \simeq 5.7\gamma$. We can also calculate the reduction in the strength of optical relaxation influencing the dynamics of the slow-light soliton. According to our solution Eq.(18), the amplitude of the soliton decays with the rate 2α. Therefore, we can introduce an effective relaxation constant defined as follows (see Eq.(17))

$$\gamma^* = \frac{\gamma}{p_0^2 + 1}. \quad (31)$$

In the case considered above, $\gamma^* \simeq \gamma/340$. We note that the effective value of the relaxation constant γ is significantly lower for the soliton than for an arbitrary pulse. So, the spontaneous emission is greatly suppressed for soliton pulses due to the nonlinear interaction with the medium. The effective optical relaxation time τ^*_{rel} is larger than the pulse length and approximately equals $2.2\, t_p$. For the pulse delay $\Delta t = 7.05\,\mu s$ reported by Hau [2] we calculated the decay of the soliton pulse amplitude at the maximum and compared it with a reference pulse Ω^*_a propagating in the system without relaxation. We obtained

$$\frac{\Omega_a(\Delta\tau + t_p/2, \zeta_c)}{\Omega^*_a} \simeq 0.2$$

The reference pulse Ω^*_a is modeled traveling under the same conditions in the system without relaxation. We added a half of the pulse time width, because it started interacting with the medium approximately $t_p/2$ microseconds earlier than the maximum entered inside Fig. 1. This result agrees very well with the measurements reported by Hau. We also note that the distance that the slow-light soliton propagated in the medium during the time $\Delta\tau + t_p$ is approximately equal $200\mu m$, which is of order of the atomic cloud in the experiment [2]. We need to emphasize that in the presence of the relaxation the velocity of the soliton is not a constant any more like in the ideal case (see Eq.(20). We calculated that the average value of the velocity is approximately $22m/s$ compared to experimentally measured $32m/s$ for the Gaussian pulse.

A more important feature of the dynamics is in the fact that the soliton spatial form does not depend on the relaxation at all. The soliton shape remains unchanged along the propagation inside the medium (see Fig. 1). This fact plays a crucial role for storage and retrieval of optical information in potential applications.

V. DISCUSSION

The results of our work remain valid beyond the constraints of the transparency window defined in the linear theory. We demonstrated that due to strong nonlinear interaction between the probe and controlling fields it is possible to preserve the spacial shape of the optical signals even in the presence of strong optical relaxation. The comparison of our theory and experimental results [2, 3] shows a very good agreement. We provided rigorous analytical estimates for the largest distance that the slow-light soliton can propagate in the medium with relaxation and also described the dependence of the soliton velocity on the relaxation constant γ.

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